

# ONE-DIMENSIONAL POLYNOMIAL MAPS, PERIODIC POINTS AND MULTIPLIERS

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ABSTRACT. We discuss tangent maps related to the multipliers of periodic points of a typical one-dimensional polynomial map.

MSC: 14A25; 37F10

## 1. DEFINITIONS, NOTATION, STATEMENTS

We write  $\mathbb{C}$  for the field of complex numbers. For every positive integer  $m$  let us consider the affine space  $\mathbf{A}^m = \mathbb{C}^m$  of all monic complex polynomials of degree  $m$

$$u(x) = x^m + \sum_{i=0}^{m-1} a_i x^i$$

with coefficients  $a = (a_0, \dots, a_{m-1}) \in \mathbb{C}^m = \mathbf{A}^m$ . It is convenient to identify the tangent space  $\mathbb{C}^m$  to  $u(x) \in \mathbf{A}^m$  with the space of all polynomials  $p(x)$  of degree  $\leq m-1$ . Namely, to a polynomial  $p(x) = \sum_{i=0}^{m-1} c_i x^i$  one assigns the tangent vector  $(c_0, \dots, c_{m-1}) \in \mathbb{C}^m$  that corresponds to “the tangency class at  $u(x)$  of the curve”  $\epsilon \rightarrow u(x) + \epsilon \cdot p(x) \in \mathbf{A}^m$  [7, Part II, Ch. III, Sect. 8, pp. 81–82].

Let  $P_m \subset \mathbf{A}^m$  be the everywhere dense Zariski-open affine variety that consists of all polynomials without multiple roots. Let  $f(x) \in P_m$  and let us choose a root  $\alpha$  of  $f(x)$ . Locally (with respect to  $a$ ), one may view  $\alpha$  (using Implicit Function Theorem) as a holomorphic (univalued) function in  $a = (a_0, \dots, a_{m-1})$ . We have ([10, Sect. 2])

$$d\alpha/da_i = -[f'(\alpha)]^{-1} \alpha^i.$$

(Since  $\alpha$  is a simple root of  $f(x)$ , we have  $f'(\alpha) \neq 0$ .) We also have (ibid)

$$df'(\alpha)/da_i = i\alpha^{i-1} - [f'(\alpha)]^{-1} \alpha^i f''(\alpha)$$

(of course, if  $i = 0$  then the first term disappears). Using these formulas, let us compute the differential  $dN : \mathbb{C}^m \rightarrow \mathbb{C}$  (at  $f(x)$ ) of locally defined holomorphic function

$$N : P_m \rightarrow \mathbb{C}, \quad f(x) \mapsto f'(\alpha).$$

It follows that  $dN$  sends the tangent vector  $p(x) = \sum_{i=0}^{m-1} c_i x^i$  to the number

$$dN(p(x)) = \sum_{i=0}^{m-1} c_i \frac{df'}{da_i}(\alpha) = p'(\alpha) - [f'(\alpha)]^{-1} p(\alpha) f''(\alpha).$$

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**Example 1.1.** Suppose that  $m \geq 3$  and  $f(x) = x^m - x$ . Then  $\alpha$  is either zero or  $(m-1)$ th root of unity. If  $\alpha = 0$  then  $f''(0) = 0$  and

$$dN(p(x)) = p'(0) = c_1.$$

The *gradient* of  $N$  at  $f(x) = x^m - x$  (with respect to the root 0) is

$$Q_1(0) = (0, 1, \dots, 0) \in \mathbb{C}^n.$$

If  $\alpha^{m-1} = 1$  then

$$\begin{aligned} f'(\alpha) &= m\alpha^{m-1} - 1 = m - 1, \\ f''(\alpha) &= m(m-1)\alpha^{m-2} = m(m-1)/\alpha, \end{aligned}$$

and

$$dN(p(x)) = p'(\alpha) - \frac{mp(\alpha)}{\alpha}.$$

The *gradient* of  $N$  at  $f(x) = x^m - x$  (with respect to the root  $\alpha$ ) is

$$Q_1(\alpha) = \left(-\frac{m}{\alpha}, (1-m), (2-m)\alpha, \dots, -\alpha^{m-2}\right) \in \mathbb{C}^n.$$

Let  $n \geq 2$  be an integer and  $g(x) \in \mathbb{C}[x]$  a degree  $n$  monic polynomial with complex coefficients. For every positive integer  $r$  we denote by  $g^{\circ r}(x)$  the composition  $g(\dots g(x))$  ( $r$  times). Clearly,  $g^{\circ r}(x)$  is a degree  $n^r$  monic polynomial with complex coefficients. Let us consider the polynomial map

$$G : \mathbb{C} \rightarrow \mathbb{C}, \quad z \mapsto g(z).$$

Clearly, the fixed points of  $G$  are exactly the roots of  $g(x) - x$  while the roots of  $g^{\circ r}(x) - x$  are exactly the points of period (dividing)  $r$ .

**Example 1.2.** If  $g(x) = x^n$  then  $g^{\circ r}(x) = x^{n^r}$ ,  $g^{\circ r}(x) - x = x^{n^r} - x$ .

We write  $Z_{n,r} \subset \mathbf{A}^n$  for the everywhere dense Zariski-open affine subvariety that consists of all monic degree  $n$  polynomials  $g(x)$  such that  $g^{\circ r}(x) - x$  lies in  $P_{n^r}$ . For example  $x^n \in Z_{n,r}$  for all  $r$ . Clearly,

$$Z_{n,1} = \{f(x) + x \mid f(x) \in P_n\}.$$

It is also clear that the holomorphic map

$$U_n : Z_{n,1} \rightarrow P_n, \quad g(x) \mapsto g(x) - x$$

is a holomorphic isomorphism, whose tangent map

$$dU_n : \mathbb{C}^n \rightarrow \mathbb{C}^n$$

is the identity map at all points of  $Z_{n,1}$ .

Let us consider a locally defined holomorphic function

$$M^r : Z_{n,r} \rightarrow Z_{n^r,1} \rightarrow P_{n^r} \rightarrow \mathbb{C}, \quad g(x) \mapsto g^{\circ r}(x) \xrightarrow{U_{n^r}} g^{\circ r}(x) - x \xrightarrow{N} [g^{\circ r}(x) - x]'(\alpha)$$

where  $\alpha$  is a root of  $g^{\circ r}(x) - x$ . We are going to discuss its differential

$$dM^r_{|g(x)} : \mathbb{C}^n \rightarrow \mathbb{C},$$

paying special attention to the computation of the corresponding gradient

$$\text{grad}(M^r)_{|g(x)} \in \mathbb{C}^n$$

at the point  $g(x) = x^n \in Z_{n,r}$ . In what follows we denote this gradient by  $Q_r(\alpha)$ . This notation is compatible with our previous notation for  $Q_1(\alpha)$  in Example 1.1.

**Remark 1.3.** Let us consider the locally defined multiplier function

$$\text{Mult}^r : Z_{n,r} \rightarrow Z_{n^r,1} \rightarrow \mathbb{C}, \quad g(x) \mapsto [g^{\circ r}]'(\alpha).$$

Clearly,  $M^r(g) = \text{Mult}^r(g) - 1$ . It follows that the differentials  $dM^r$  and  $d\text{Mult}^r$  everywhere coincide. In other words

$$\text{grad}(M^r)|_{g(x)} = \text{grad}(\text{Mult}^r)|_{g(x)} \quad \forall g(x) \in Z_{n,r}.$$

**1.4.** In order to state our main results, let us consider a positive integer  $\ell$  and a sequence  $\{r_1, \dots, r_\ell\}$  of  $\ell$  positive integers. Let  $Z(n, \ell; r_1, \dots, r_\ell)$  be the intersection of all  $Z_{n, r_i}$ ; it is a nonempty Zariski-open affine subvariety in  $\mathbf{A}^n$  that contains  $g(x) = x^n$ . Let  $g(x) \in Z(n, \ell; r_1, \dots, r_\ell)$ . For each  $i$  pick a complex number  $\beta_i$  that is a periodic point of  $G : z \mapsto g(z)$  of **exact period**  $r_i$ . Locally (with respect to  $g$ ), each  $\beta_i$  is a holomorphic function (in the coefficients of  $g(x)$ .)

Suppose that  $\beta_1, \dots, \beta_\ell$  belong to **distinct orbits** of  $z \mapsto g(z)$ . Let us consider the following  $\ell$  locally defined holomorphic functions

$$\text{Mult}_{\beta_i, r_i} : Z(n, \ell; r_1, \dots, r_\ell) \rightarrow \mathbb{C}, \quad g(x) \mapsto [g^{\circ r_i}]'(\beta_i).$$

Let  $Z^0(n, \ell; r_1, \dots, r_\ell)$  be the set of all polynomials  $g(x) \in Z(n, \ell; r_1, \dots, r_\ell)$  such that the  $\ell$ -element set

$$\{\text{grad}(\text{Mult}_{\beta_i, r_i})|_{g(x)} \in \mathbb{C}^n \mid 1 \leq i \leq \ell\}$$

of gradients of  $\text{Mult}_{\beta_i, r_i}$ 's at  $g(x)$  is linearly independent in  $\mathbb{C}^n$  for every choice of  $\{\beta_1, \dots, \beta_\ell\}$ . Clearly,  $Z^0(n, \ell; r_1, \dots, r_\ell)$  is an open subset of  $Z(n, \ell; r_1, \dots, r_\ell)$  and therefore of  $\mathbf{A}^n$  in complex topology. However, this set may be empty; e.g., when  $\ell \geq n$ .

The following statements are main results of this paper.

**Theorem 1.5.** *The set  $Z^0(n, \ell; r_1, \dots, r_\ell)$  is a Zariski-open subset of  $Z(n, \ell; r_1, \dots, r_\ell)$  and therefore of  $\mathbf{A}^n$ .*

**Theorem 1.6.** *Suppose that  $n \geq 3$ . Assume that  $\sum_{i=1}^\ell r_i \leq n$ . If  $r_j = 1$  for some  $j$  then we assume additionally that  $\sum_{i=1}^\ell r_i < n$ .*

*Then  $Z^0(n, \ell; r_1, \dots, r_\ell)$  contains  $g(x) = x^n$  and therefore is nonempty.*

Combining Theorems 1.5 and 1.6, we obtain the following statement.

**Corollary 1.7.** *Suppose that  $n \geq 3$ . Assume that  $\sum_{i=1}^\ell r_i \leq n$ . If  $r_j = 1$  for some  $j$  then we assume additionally that  $\sum_{i=1}^\ell r_i < n$ .*

*Then  $Z^0(n, \ell; r_1, \dots, r_\ell)$  is a Zariski-open everywhere dense subset of  $Z(n, \ell; r_1, \dots, r_\ell)$  that contains  $g(x) = x^n$ .*

**Example 1.8.** Suppose that  $\ell = n - 1$  and all  $r_i = 1$  (i.e., all the  $\beta_i$  involved are fixed points). It follows from results of [10, 6] that

$$Z^0(n, n-1; 1, \dots, 1) = Z(n, n-1; 1, \dots, 1) = Z_{n,1}.$$

**Remark 1.9.** In the notation and assumptions of Corollary 1.7, let us consider the locally defined holomorphic map

$$Z(n, \ell; r_1, \dots, r_\ell) \rightarrow \mathbb{C}^\ell$$

defined by the collection of functions  $\{\text{Mult}_{\beta_i, r_i}\}_{i=1}^\ell$ . Corollary 1.7 asserts that this map has (maximal) rank  $\ell$  on a nonempty Zariski-open subvariety

$$Z^0(n, \ell; r_1, \dots, r_\ell) \subset Z(n, \ell; r_1, \dots, r_\ell)$$

for every choice of periodic points  $\{\beta_1, \dots, \beta_\ell\}$ . It would be interesting to study its image. For the case of fixed points (i.e., when all  $r_i = 1$ ), see [3].

Notice that Remark 1.9) gives a partial answer to a question of Yu.S. Ilyashenko, who was interested in the case of two orbits, in connection with [1, 2].

**Remark 1.10.** In the case of two orbits, it turns out (see Examples 2.3, 2.4 and Remark 2.5 below) that  $g(x) = x^n$  does not belong to  $Z(n, 2; r_1, r_2)$  if either  $r_1 = r_2 = n - 1$  or  $r_1 = 1, r_2 = n - 1$ . It would be interesting to find out whether in these cases  $Z(n, 2; r_1, r_2)$  is empty.

The paper is organized as follows. In Section 2 we compute explicitly the differentials  $d\text{Mult}_{\beta, r}$  at  $g(x) = x^n$  where  $\beta$  is a  $(n^r - 1)$ th root of unity. This allows us to write down explicitly the corresponding gradients  $Q_r(\beta) \in \mathbb{C}^n$ . Now Theorem 1.6 becomes equivalent to an assertion that the corresponding set of vectors  $\{Q_{r_i}(\beta_i)\}$  (and  $Q_1(0)$  if one of  $r_i$  is 1) is linearly independent in  $\mathbb{C}^n$ . We prove this assertion in Section 3. Using standard properties of finite maps [8, Ch. 1], we prove Theorem 1.5 in Section 4.

## 2. COMPUTATIONS OF TANGENT MAPS

**Lemma 2.1.** *Let us consider the holomorphic map  $\Phi_{n, r} : \mathbf{A}^n \rightarrow \mathbf{A}^{n^r}$  that sends a degree  $n$  monic polynomial  $g(x)$  to the monic degree  $n^r$  polynomial  $g^{\circ r}(x)$ . Then the tangent map  $d\Phi_{n, r}$  at  $g(x) = x^n$  is as follows. It sends a tangent vector  $x^k$  (at the point  $x^n$ ) to the tangent vector*

$$p_{r, k}(x) := \sum_{i=1}^r n^{r-i} x^{n^r - n^i + n^{i-1}k}$$

(at the point  $x^{n^r}$ ). In particular,

$$p_{r, 0}(x) := \sum_{i=1}^r n^{r-i} x^{n^r - n^i} = n^{r-1} x^{n^r - n} + n^{r-2} x^{n^r - n^2} + \dots,$$

$$p_{r, 1}(x) := \sum_{i=1}^r n^{r-i} x^{n^r - n^i + n^{i-1}} = n^{r-1} x^{n^r - n + 1} + n^{r-2} x^{n^r - n^2 + n} + \dots$$

and

$$\deg(p_{r, 0}) = n^r - n, \deg(p_{r, 1}) = n^r - n + 1, \deg(p_{r, k}) = n^r - n + k.$$

*Proof.* Notice that  $p_{1, k}(x) = x^k$  and for all positive integers  $r$

$$p_{r+1, k}(x) = nx^{(n-1)n^r} p_{r, k}(x) + x^{kn^r}.$$

*Induction by  $r$ .* We need to prove that if  $g^{[\epsilon]}(x) = x^n + \epsilon x^k$  then  $[g^{[\epsilon]}]^{\circ r}(x) = x^{n^r} + \epsilon p_{r, k}(x) + O(\epsilon^2)$ . If  $r = 1$  then it is obvious. Assume that this assertion is true for  $r$  and let us check it for  $r + 1$ . We have

$$\begin{aligned} [g^{[\epsilon]}]^{\circ(r+1)}(x) &= \{[g^{[\epsilon]}]^{\circ r}(x)\}^n + \epsilon \{[g^{[\epsilon]}]^{\circ r}(x)\}^k = (x^{n^r} + \epsilon p_{r, k}(x) + O(\epsilon^2))^n + \epsilon (x^{n^r} + \epsilon p_{r, k}(x) + O(\epsilon^2))^k = \\ &= x^{n^{r+1}} + \epsilon n x^{(n-1)n^r} p_{r, k}(x) + \epsilon x^{kn^r} + O(\epsilon^2) = \\ &= x^{n^{r+1}} + \epsilon \{n x^{(n-1)n^r} p_{r, k}(x) + x^{kn^r}\} + O(\epsilon^2) = x^{n^{r+1}} + \epsilon p_{r+1, k}(x) + O(\epsilon^2). \end{aligned}$$

□

Since all  $p_{r,k}$  (for given  $n$  and  $r$ ) have distinct degrees, the set  $\{p_{r,0}, \dots, p_{r,n-1}\}$  is linearly independent. This means that the rank of the tangent map to  $\Phi_{n,r}$  at  $g(x) = x^n$  is  $n$ , i.e. the tangent map at this point is injective and its image coincides with

$$\oplus_{i=0}^{n-1} \mathbb{C} \cdot p_{r,k}.$$

**2.2.** Suppose that  $n^r \geq 3$ . Let us compute the differential

$$d\text{Mult}^r = dM^r = d(M\Phi_{n,r}) = dM \circ d\Phi_{n,r}$$

at  $g(x) = x^m \in Z_{n,r}$ . Clearly,

$$\Phi_{n,r}(x^n) = x^{n^r} \in P_m$$

with  $m = n^r$ . Let  $\alpha$  be a nonzero root of  $x^m - x$ , i.e.,  $\alpha^{n^r-1} = 1$ . Using Lemma 2.1 and Example 1.1, we obtain the following. The image

$$q_{r,k}(\alpha) := d\text{Mult}_{|g(x)=x^n}^r(x^k)$$

of tangent vector  $x^k$  to  $g(x) = x^n \in Z_{n,r}$  is

$$\begin{aligned} p'_{r,k}(\alpha) - \frac{n^r p_{r,k}(\alpha)}{\alpha} &= \\ \alpha^{-1} \sum_{i=1}^r (n^r - n^i + n^{i-1}k) n^{r-i} \alpha^{n^r - n^i + n^{i-1}k} - \alpha^{-1} \sum_{i=1}^r n^r n^{r-i} \alpha^{n^r - n^i + n^{i-1}k} &= \\ \alpha^{n^r-1} \sum_{i=1}^r [n^{2r-i} - (n^{2r-i} - n^r + kn^{r-1})] \alpha^{-n^i + n^{i-1}k} &= (n^r - kn^{r-1}) \sum_{i=1}^r \alpha^{-n^i + n^{i-1}k} = \\ (n-k)n^{r-1} \sum_{i=1}^r \alpha^{-n^i + n^{i-1}k} &= (n-k)n^{r-1} \sum_{i=1}^r \left( \frac{1}{\alpha^{n^{i-1}}} \right)^{n-k}. \end{aligned}$$

In other words,

$$q_{r,k}(\alpha) = d\text{Mult}_{|g(x)=x^n}^r(x^k) = (n-k)n^{r-1} \sum_{i=1}^r \left( 1/\alpha^{n^{i-1}} \right)^{n-k}.$$

Notice that

$$q_{r,k}(\alpha) = q_{r,k}(\alpha^n), \quad q_{r,0}(\alpha) = nq_{r,n-1}(\alpha).$$

It follows that the *gradient* of  $\text{Mult}^r$  at  $g(x) = x^n$  (with respect to  $\alpha$ ) is

$$Q_r(\alpha) = (q_{r,0}(\alpha), q_{r,1}(\alpha), \dots, q_{r,n-1}(\alpha)) = (nq_{r,n-1}(\alpha), q_{r,1}(\alpha), \dots, q_{r,n-1}(\alpha)) \in \mathbb{C}^n.$$

Clearly,

$$Q_r(\alpha) = Q_r(\alpha^n) = \dots = Q_r(\alpha^{n^{r-1}}).$$

Let  $\mathcal{O}(\alpha) = \{\alpha, \alpha^n, \dots, \alpha^{n^{r-1}}\}$  be the orbit of  $\alpha$  with respect to  $z \mapsto z^n$  and let  $d(\alpha)$  be the cardinality of the set  $\mathcal{O}(\alpha)$ . Clearly,  $d(\alpha)$  is a positive integer that divides  $r$  and

$$\beta^{d(\alpha)} = \beta \quad \forall \beta \in \mathcal{O}(\alpha).$$

It is also clear that

$$q_{r,k}(\alpha) = \frac{r}{d(\alpha)} (n-k)n^{r-1} \sum_{\beta \in \mathcal{O}(\alpha)} (1/\beta)^{n-k}.$$

This implies that

$$Q_r(\alpha) = \frac{rn^r}{d(\alpha)n^{d(\alpha)}} \cdot Q_{d(\alpha)}(\alpha).$$

**Example 2.3.** Suppose that  $n = 3$  and  $r = 2$ . Then  $n^r - 1 = 8$ . Let  $\alpha$  be a 8th root of unity that is not  $\pm 1$ . Then  $\alpha$  is a periodic point of exact period 2 for  $z \mapsto z^3$ . We have

$$Q_2(\alpha) = 3^1 \cdot \left( 3 \left[ \frac{1}{\alpha} + \frac{1}{\alpha^3} \right], 2 \left[ \frac{1}{\alpha^2} + \frac{1}{\alpha^6} \right], \left[ \frac{1}{\alpha} + \frac{1}{\alpha^3} \right] \right).$$

So, if  $\alpha$  is a primitive fourth root of unity, i.e.,

$$\alpha^2 = -1, \alpha = \pm \mathbf{i},$$

then

$$\frac{1}{\alpha} + \frac{1}{\alpha^3} = 0, \frac{1}{\alpha^2} + \frac{1}{\alpha^6} = -2$$

and

$$Q_2(\alpha) = 3 \cdot (0, -2, 0) = (0, -6, 0).$$

However, both  $\mathbf{i}$  and  $-\mathbf{i}$  lie in the same orbit.

If  $\alpha$  is a primitive 8th root of unity then

$$\alpha^4 = -1, 1 + \frac{1}{\alpha^4} = 0$$

and therefore

$$Q_2(\alpha) = 3 \cdot \left( 3 \left[ \frac{1}{\alpha} + \frac{1}{\alpha^3} \right], 0, \left[ \frac{1}{\alpha} + \frac{1}{\alpha^3} \right] \right) = \frac{3}{\alpha^3} \cdot (3[\alpha^2 + 1], 0, \alpha^2 + 1) = \frac{3(\alpha^2 + 1)}{\alpha^3} \cdot (3, 0, 1).$$

Now if we put  $\beta = \alpha^{-1}$  then  $\alpha$  and  $\beta$  are primitive 8th roots of unity that do not belong to the same orbit while  $Q_2(\alpha)$  and  $Q_2(\beta)$  generate the same line  $\mathbb{C} \cdot (3, 0, 1)$  in  $\mathbb{C}^3$ . This implies that  $Z(3, 2; 2, 2)$  does *not* contain  $g(x) = x^3$ .

**Example 2.4.** Suppose that  $n = r + 1$  and  $r \geq 2$ . Then for all positive integers  $i$

$$n^i = (1 + r)^i = 1 + i \cdot r^1 + \dots + \binom{i}{j} r^j + \dots + r^i.$$

It follows that  $n^i$  is congruent to  $1 + ir$  modulo  $r^2$ . In particular,  $n^r - 1$  is divisible by  $r^2 = (n - 1)^2$ . It also follows that  $(n^i - n)/(n - 1) = n^{i-1}$  is congruent to  $i - 1$  modulo  $r$  and therefore

$$n^i - n \equiv (i - 1)r \pmod{r^2}.$$

Suppose that  $\alpha$  is a *primitive*  $r^2$ th root of unity. Then  $\alpha^{n^r - 1} = 1$  and therefore

$$\alpha^{n^r} = \alpha,$$

i.e.,  $\alpha$  is a periodic point for the map  $z \rightarrow z^n$ . Clearly, its period divides  $r$ . On the other hand, for all positive integers  $i < r$  the power  $n^i$  is *not* congruent to 1 modulo  $r^2$  and therefore  $\alpha^{n^i} \neq \alpha$ . It follows that  $\alpha$  has exact period  $r$ .

The number  $\gamma := \alpha^{1-n} = \alpha^{-r}$  is a primitive  $r$ th root of unity. For each integer  $k$  with  $0 \leq k \leq n - 1$  the number  $\delta = \gamma^{n-k}$  is an  $r$ th root of unity. Clearly,  $\delta \neq 1$  if and only if  $n - k \neq n - 1$ , i.e.  $k \neq 1$ . In particular, if  $k \neq 1$  then  $\sum_{i=1}^r \delta^i = 0$ .

We have

$$\begin{aligned} q_{r,k}(\alpha) &= (n - k)n^{r-1} \sum_{i=1}^r \left( \alpha^{-n^i} \right)^{n-k} = \\ &= (n - k)n^{r-1} \cdot \alpha^{n(k-n)} \sum_{i=1}^r \left( \alpha^{n-n^i} \right)^{n-k} = (n - k)n^{r-1} \alpha^{n(k-n)} \sum_{i=1}^r \delta^{i-1} = \end{aligned}$$

$$(n-k)n^{r-1}\delta^{-1}\alpha^{n(k-n)}\sum_{i=1}^r\delta^i=0$$

if  $k \neq 1$ . On the other hand, if  $k = 1$  then  $\delta = 1$  and  $q_{r,1}(\alpha) = r(n-1)n^{r-1}\gamma$ . It follows that

$$Q_r(\alpha) = (0, r(n-1)n^{r-1}\gamma, 0, \dots, 0) = r(n-1)n^{r-1}\gamma \cdot Q_1(0) \in \mathbb{C}^n.$$

This implies that  $Z^0(r+1, 2; r, 1)$  does not contain  $g(x) = x^n$ .

**Remark 2.5.** Suppose that  $r > 2$  and  $n = r + 1$ . Example 2.4 tells us that if  $\alpha$  and  $\beta$  are primitive  $r^2$ th roots of unity then  $Q_r(\alpha)$  and  $Q_r(\beta)$  generate the same line  $\mathbb{C} \cdot (0, 1, \dots, 0)$  in  $\mathbb{C}^n$ . Since  $r > 2$ , the number  $\varphi(r^2)$  of primitive  $r^2$ th roots of unity is strictly greater than  $r$ . (Here  $\varphi$  is the Euler function.) In particular, we may choose such  $\alpha$  and  $\beta$  from different orbits (of length  $r$ ) of the map  $z \mapsto z^n$ . It follows that  $Z^0(r+1, 2; r, r)$  does not contain  $g(x) = x^n$ .

### 3. LINEAR INDEPENDENCE

As was already pointed out, Theorem 1.6 is an immediate corollary of the following statement.

**Theorem 3.1.** *Let  $\ell$  be a positive integer. Let  $\{r_1, \dots, r_\ell\}$  be a sequence of  $\ell$  positive integers. Let  $\{\alpha_1, \dots, \alpha_\ell\}$  be a sequence of distinct complex numbers such that*

$$\alpha_i^{n^{r_i}-1} = 1 \quad \forall i = 1, \dots, \ell.$$

*Assume that  $\{\alpha_1, \dots, \alpha_\ell\}$  belong to different orbits of the map  $z \rightarrow z^n$ . Then:*

- (i) *the set of  $\ell$  vectors  $\{Q_{r_1}(\alpha_1), \dots, Q_{r_\ell}(\alpha_\ell)\}$  in  $\mathbb{C}^n$  is linearly independent if  $n \geq \sum_{j=1}^\ell d(\alpha_j)$ . In particular, if  $\sum_{i=1}^\ell r_i \leq n$  then the  $\ell$ -tuple  $\{Q_{r_1}(\alpha_1), \dots, Q_{r_\ell}(\alpha_\ell)\}$  is linearly independent in  $\mathbb{C}^n$ .*
- (ii) *If  $n \geq 2 + \sum_{j=1}^\ell d(\alpha_j)$  then the  $(\ell+1)$ -tuple  $\{Q_1(0); Q_{r_1}(\alpha_1), \dots, Q_{r_\ell}(\alpha_\ell)\}$  is a linearly independent set in  $\mathbb{C}^n$ . In particular, if  $\sum_{i=1}^\ell r_i < n-1$  then the  $(\ell+1)$ -tuple  $\{Q_1(0); Q_{r_1}(\alpha_1), \dots, Q_{r_\ell}(\alpha_\ell)\}$  is a linearly independent set in  $\mathbb{C}^n$ .*

*Proof of Theorem 3.1.* In the course of the proof we will use the following elementary statement that will be proven at the end of this section.

**Lemma 3.2.** *Let  $d$  be a positive integer,  $S$  a set of  $d$  nonzero complex numbers. Let  $c : S \rightarrow \mathbb{C}$  be a function such that for all positive integers  $u = 1, \dots, d$*

$$\sum_{\beta \in S} \frac{c(\beta)}{\beta^u} = 0.$$

*Then  $c(\beta) = 0$  for all  $\beta \in S$ .*

Let us continue to prove Theorem 3.1. Replacing each  $r_i$  by  $d(\alpha_i)$  we may and will assume that  $r_i = d(\alpha_i)$ , i.e., the orbit  $\mathcal{O}(\alpha_i)$  of  $\alpha_i$  consists of  $r_i$  distinct elements (for all  $i$ ). We also assume that  $n \geq \sum_{i=1}^\ell r_i$ .

Let  $\{c_1, \dots, c_\ell\}$  be a sequence of  $\ell$  complex numbers such that

$$\sum_{i=1}^\ell c_i Q_{r_i}(\alpha_i) = 0.$$

Let  $S$  be the (disjoint) union of all  $\mathcal{O}(\alpha_i)$ , which consists of  $\left(\sum_{i=1}^{\ell} r_i\right)$  elements. Let us define a complex valued function  $c$  on  $S$  that assigns to  $\alpha \in \mathcal{O}(\alpha_i)$  the complex number

$$c(\alpha) := n^{r_i-1} c_i.$$

Then we obtain for all  $k = 0, 1, \dots, n-1$

$$0 = \sum_{i=1}^{\ell} c_i q_{r_i, k}(\alpha_i) = (n-k) \sum_{\alpha \in S} c(\alpha) (1/\alpha)^{n-k}.$$

This implies that

$$\sum_{\alpha \in S} c(\alpha) (1/\alpha)^u = 0$$

for all positive integers  $u = 1, \dots, n$ .

It follows from Lemma 3.2 applied to  $d = \sum_{i=1}^{\ell} r_i$  that all  $c(\alpha) = 0$ . Since all  $n^{r_i-1} \neq 0$ , we conclude that all  $c_i = 0$ . This proves (i).

Now assume that  $n \geq 2 + \sum_{i=1}^{\ell} r_i$ . We are going to prove (ii). Let  $\{c_0, c_1, \dots, c_{\ell}\}$  be a sequence of  $\ell$  complex numbers such that

$$c_0 Q_1(0) + \sum_{i=1}^{\ell} c_i Q_{r_i}(\alpha_i) = 0.$$

We have

$$-c_0 Q_1(0) = \sum_{i=1}^{\ell} c_i Q_{r_i}(\alpha_i).$$

Recall that all the coordinates of  $Q_1(0)$  except the second one do vanish. This implies that

$$0 = \sum_{i=1}^{\ell} c_i q_{r_i, k}(\alpha_i) = (n-k) \sum_{\alpha \in S} c(\alpha) (1/\alpha)^{n-k}$$

for all  $k = 0, \dots, n-1$  except  $k = 1$ . It follows that that

$$\sum_{\alpha \in S} c(\alpha) (1/\alpha)^u = 0$$

for all positive integers  $u = 1, \dots, n-2$ . Since  $n-2 \geq d$ , the same arguments with Lemma 3.2 as above prove that  $c_i = 0$  for all positive integers  $i$  and therefore  $-c_0 Q_1(0) = 0$ , i.e.,  $c_0 = 0$ .  $\square$

*Proof of Lemma 3.2.* This Lemma is a variant of well-known classical results (e.g., see [9]). Let us consider the rational function

$$X(t) = \sum_{\beta \in S} \frac{c(\beta)}{\beta - t} = \sum_{\beta \in S} \frac{c(\beta)/\beta}{1 - \frac{t}{\beta}}.$$

Clearly,

$$X(t) = \frac{Q(t)}{\prod_{\beta \in S} (\beta - t)}$$

where  $Q(t)$  is a polynomial, whose degree does not exceed  $d-1$ . (Recall that  $d = \#(S)$ .) For each positive integer  $u$ , the number  $\sum_{\beta \in S} c(\beta)/\beta^u$  is the  $(u-1)$ th coefficient of the Taylor power series of  $X(t)$  at the origin (see [4, Ch. 1, Sect. 2]). It follows that  $X(t)$  has a zero of order  $\geq d$  at the origin. This implies that  $Q(t)$  is



divisible by  $t^d$  and therefore  $Q(t) = 0$ , i.e.  $X(t) = 0$ . Since  $-c(\beta)$  is the residue of  $X(t)$  at  $t = \beta$  for all  $\beta \in S$ , we conclude that  $c(\beta) = 0$ .  $\square$

**Remark 3.3.** One may give even more elementary proof of Lemma 3.2, using the nondegeneracy of the Vandermonde matrix of size  $d \times d$  for *distinct* numbers  $\{1/\beta \mid \beta \in S\}$ .

#### 4. OPENNESS IN ZARISKI TOPOLOGY

The aim of this Section is to prove Theorem 1.5. We will need the following well known easy statement.

**Lemma 4.1.** *Let  $n \geq 2$  be an integer and  $g(x) \in \mathbb{C}[x]$  is a monic degree  $n$  polynomial. Suppose that  $g(x) - x$  has a multiple root say,  $\alpha$ . Then for all positive integers  $r$  the complex number  $\alpha$  is a multiple root of  $g^{\circ r}(x) - x$ .*

*Proof.* We have

$$g(\alpha) = \alpha, \quad g'(\alpha) = 1.$$

It follows easily that

$$g^{\circ r}(\alpha) = \alpha, \quad [g^{\circ r}]'(\alpha) = 1.$$

This means that

$$[g^{\circ r}(x) - x](\alpha) = 0, \quad [g^{\circ r}(x) - x]'(\alpha) = 0.$$

In other words,  $\alpha$  is a multiple root of  $g^{\circ r}(x) - x$ .  $\square$

**Corollary 4.2.** *Let  $m$  be a positive integer that divides  $r$ . Suppose that  $g^{\circ m}(x) - x$  has a multiple root say,  $\alpha$ . Then  $\alpha$  is a multiple root of  $g^{\circ r}(x) - x$ .*

Let  $g(x) \in Z_{n,r}$ . If  $m$  is a positive divisor of  $r$  then Corollary 4.2 implies that  $g(x) \in Z_{n,m}$ . The number of periodic points of exact period  $m$  for  $z \mapsto g(z)$  is

$$\nu_n(m) = \sum_{m' \mid m} \mu\left(\frac{m}{m'}\right) n^{m'}$$

where  $\mu$  is the Möbius function [5, pp. 74–75]. In particular, the number of orbits of length  $m$  equals

$$d(n, m) = \frac{\nu_n(m)}{m}.$$

For each positive divisor  $m$  of  $r$  we pick a  $d(n, m)$ -element set  $S_m$  and consider the corresponding  $d(n, m)$ -dimensional coordinate space  $\mathbb{C}^{S_m}$  of all  $\mathbb{C}$ -valued functions on  $S_m$ .

Let us consider the Zariski-closed subvariety

$$\hat{Z}_{n,r} \subset Z_{n,r} \times \prod_{m \mid r} \mathbb{C}^{S_m}$$

that is cut out by the following equations imposed on

$$\{g; \phi_m : S_m \rightarrow \mathbb{C}, \quad m \mid r\} \in Z_{n,r} \times \prod_{m \mid r} \mathbb{C}^{S_m}.$$

For each  $s \in S_m$  the complex number  $\phi_m(s)$  is a periodic point, whose period divides  $m$ , with respect to  $z \mapsto g(z)$ , i.e.  $g^{om}(\phi_m(s)) = \phi_m(s)$ . In addition, we require that

$$g^{or}(x) - x = \prod_{m|r} \left( \prod_{s \in S_m} (x - \phi_m(s)) \prod_{i=1}^{m-1} (x - g^{oi}(\phi_m(s))) \right).$$

In other words, the coefficients of both polynomials in  $x$  do coincide.

Recall that  $g^{or}(x) - x$  has no multiple roots. It follows that each map  $\phi_m : S_m \rightarrow \mathbb{C}$  is injective, its image consists of elements of exact period  $m$  while distinct elements of  $S_m$  go under  $\phi_m$  to distinct orbits of length  $m$ ; in addition, every orbit of length  $m$  contains exactly one element of  $\phi_m(S_m)$ . On the other hand, for any choice of an element  $\zeta$  in every orbit of length  $m$  (for each divisor  $m$  of  $r$ ) there is (exactly one) point of  $\hat{Z}_{n,r}$  that lies above  $g(x)$  and such that the corresponding  $\phi_m(S_m)$  consists of these  $\zeta$ .

Clearly, the projection map  $\hat{Z}_{n,r} \rightarrow Z_{n,r}$  is surjective. It is also clear that it is finite. Now one may “lift”  $\text{Mult}^r = \text{Mult}_{r,\beta}$  to globally defined functions on  $\hat{Z}_{n,r}$ . Namely, for each  $s \in S_r$  the function

$$\overline{\text{Mult}}_{r,s} : \hat{Z}_{n,r} \rightarrow \mathbb{C}, \{g; \phi_m : S_m \rightarrow \mathbb{C}, m \mid r\} \mapsto \text{Mult}_{r,\phi_r(s)}(g)$$

is a globally defined regular function. If  $\phi_m(s)$  and  $\beta$  lie in the same orbit of length  $m$  then this function coincides with the composition of projection map  $\hat{Z}_{n,r} \rightarrow Z_{n,r}$  and  $\text{Mult}_{r,\beta}$ . It is also clear that the vector function

$$\overline{\text{grad}}(\text{Mult}_{r,s}) : \hat{Z}_{n,r} \rightarrow \mathbb{C}^n, \{g; \phi_m : S_m \rightarrow \mathbb{C}^n, m \mid r\} \mapsto \text{grad}(\text{Mult}_{r,\phi_r(s)})|_{g(x)}$$

is a regular map that coincides with the composition of projection map  $\hat{Z}_{n,r} \rightarrow Z_{n,r}$  and  $\text{grad}(\text{Mult}_{r,\beta}) : Z_{n,r} \rightarrow \mathbb{C}^n$  with  $\beta = \phi_r(s)$ . If  $D$  is an  $\ell$ -element subset of  $S_r$  let us consider the subset  $X_D$  of points  $v \in \hat{Z}_{n,r}$  such that the collection of  $\ell$  vectors  $\{\overline{\text{grad}}(\text{Mult}_{r,s})(v) \mid s \in D\}$  is linearly dependent in  $\mathbb{C}^n$ . Clearly,  $X_D$  is a Zariski-closed subset in  $\hat{Z}_{n,r}$ . It follows that the union  $X$  of all  $X_D$  (where  $D$  runs through all  $\ell$ -element subsets of  $S_r$ ) is also closed in  $\hat{Z}_{n,r}$ . The finiteness of the projection map implies that the image  $\bar{X}$  of  $X$  in  $Z_{n,r}$  is also Zariski-closed ([8, Ch. 1, Sect. 5.3]). On the other hand, one may easily check that  $\bar{X}$  is the complement of  $Z^0(n, \ell; r, r, \dots, r)$  in  $Z(n, \ell; r, r, \dots, r) = Z_{n,r}$ . It follows that  $Z^0(n, \ell; r, r, \dots, r)$  is Zariski-open in  $Z_{n,r}$ . This proves Theorem 1.5 in the case when  $r_1 = r_2 = \dots = r_\ell$ .

Now let us consider the general case. Let  $d$  be the number of distinct elements in the sequence  $\{r_1, \dots, r_\ell\}$  and  $R$  the corresponding  $d$ -element set of positive integers. For each  $r \in R$  we denote by  $l(r)$  the number of  $i$  with  $r_i = r$ . We write  $\hat{Z}'_{n,r}$  for the preimage of  $Z(n, \ell; r_1, r_2, \dots, r_\ell)$  in  $\hat{Z}_{n,r}$ ; clearly, the natural regular map

$$\hat{Z}'_{n,r} \rightarrow Z(n, \ell; r_1, r_2, \dots, r_\ell)$$

is finite. Let us consider the fiber product  $\hat{Z}_{n,R}$  of all  $\hat{Z}'_{n,r}$  (where  $r$  runs through  $R$ ) over  $Z(n, \ell; r_1, r_2, \dots, r_\ell)$ ; again, if we write  $\pi$  for the natural regular map

$$\hat{Z}_{n,R} \rightarrow Z(n, \ell; r_1, r_2, \dots, r_\ell)$$

then  $\pi$  is finite. We denote by  $\pi_r$  the natural finite regular map

$$\pi_r : \hat{Z}_{n,R} \rightarrow \hat{Z}'_{n,r}.$$

Now for each  $r \in R$  pick a  $l(r)$ -element subset  $D_r$  in  $S_r$  and consider the subset  $X_{D_r}$  of points  $v \in \hat{Z}_{n,R}$  such that the collection of  $(\sum_{r \in R} l(r))$  vectors

$$\{\overline{\text{grad}}(\text{Mult}_{r,s_r})(v_r) \mid s_r \in D_r, r \in R\}$$

is linearly dependent in  $\mathbb{C}^n$ . Here  $v_r = \pi_r(v) \in \hat{Z}'_{n,r}$ . Clearly,  $X_{D_r}$  is a closed algebraic subvariety in  $\hat{Z}_{n,R}$ . Now let us consider the union  $Y$  of all such  $X_{D_r}$  for all choices of  $\{D_r \mid r \in R\}$ . It is also clear that  $Y$  is also a closed algebraic subvariety in  $\hat{Z}_{n,R}$ . Since  $\pi$  is finite, the image  $\pi(Y)$  is a closed algebraic subvariety in  $Z(n, \ell; r_1, r_2, \dots, r_\ell)$ . On the other hand, one may easily check that  $Z^0(n, \ell; r_1, r_2, \dots, r_\ell)$  is the complement of  $\pi(Y)$  in  $Z(n, \ell; r_1, r_2, \dots, r_\ell)$ . It follows that  $Z(n, \ell; r_1, r_2, \dots, r_\ell)$  is Zariski-open in  $Z(n, \ell; r_1, r_2, \dots, r_\ell)$ . This ends the proof of Theorem 1.5.

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